RELATIVISTIC TRANSFORMATIONS FOR TIME SYNCHRONIZATION AND DISSEMINATION IN THE SOLAR SYSTEM

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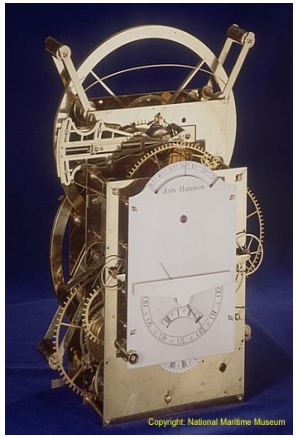
Time is essential to navigation

H1 (1730-35)

H2 (1737-40)



H3 (1740-55)

H4 (1755-59)

H5 (1772)

John Harrison solved the problem of determining longitude at sea with the invention of the marine chronometer.

A replica of H4 made by Larcum Kendall was carried by Captain James Cook in his voyage of discovery to the South Pacific in 1772.

Need for relativistic algorithms

On the surface of the Earth, time is required to determine longitude.

Similarly, for future navigation in space, a coherent time reference will be required.

For navigation at the one-meter level of precision, the time in a well defined coordinate system at a level of the order 10 ns is needed.

Mathematical algorithms taking into account the principles of general relativity are needed for consistent synchronization and dissemination of time.

The Global Positioning System (GPS) provides a paradigm for the appropriate treatment of relativity in a practical engineering system.

Proper time and coordinate time

In the theory of general relativity, there are two kinds of time.

Proper time is the actual reading of a clock. The proper times are different for clocks in different gravitational potentials and in different states of motion.

The proper time measured by a clock may be compared to the proper time measured by another clock through the intermediate variable called coordinate time.

Coordinate time, by definition, has the same value everywhere for a given event.

The relationship between proper time and coordinate time is established through the invariance of the four-dimensional space-time interval.

Invariant space-time interval

The theory of space, time, and gravitation according to the general theory of relativity is founded upon the notion of an invariant Riemannian space-time interval of the form

$$ds^2 = \sum_{\mu=0}^{3}\sum_{\nu=0}^{3} g_{\mu\nu}\, dx^\mu\, dx^\nu = g_{00}\, c^2\, dt^2 + 2\sum_{j=1}^{3} g_{0j}\, c\, dt\, dx^j + \sum_{i=1}^{3}\sum_{j=1}^{3} g_{ij}\, dx^i\, dx^j$$

For a transported clock, the space-time interval is

$$ds^2 = \sum_{\mu=0}^{3}\sum_{\nu=0}^{3} g_{\mu\nu}\, dx^\mu\, dx^\nu \equiv - c^2 d\tau^2$$

For an electromagnetic signal, the space-time interval satisfies the condition

$$ds^2 = \sum_{\mu=0}^{3}\sum_{\nu=0}^{3} g_{\mu\nu}\, dx^\mu\, dx^\nu = 0$$

Earth-orbiting satellite clocks

To a sufficient approximation in the analysis of clock transport, the components of the metric tensor in an Earth-Centered Inertial (ECI) coordinate system are

$- g_{00} \approx 1 - 2\,U / c^2$, $g_{0j} = 0$, and $g_{ij} \approx \delta_{ij}$

he elapsed coordinate time with respect to an ECI frame of reference is

$$\Delta t = \int_{\tau_0}^{\tau} \left[1 + \frac{1}{c^2} U + \frac{1}{2} \frac{1}{c^2} v^2 \right] d\tau$$

It is convenient to apply a change of scale

$$\Delta t' = \left[1 - \frac{1}{c^2} W_0 \right] \Delta t = \int_{\tau_0}^{\tau} \left[1 + \frac{1}{c^2} (U - W_0) + \frac{1}{2} \frac{1}{c^2} v^2 \right] d\tau$$

The elapsed coordinate time for an Earth-orbiting satellite clock becomes

$$\Delta t = \left[1 + \frac{3}{2} \frac{1}{c^2} \frac{GM}{a} - \frac{1}{c^2} W_0 \right] \Delta\tau + \frac{2}{c^2} \sqrt{GM\,a}\, e \sin E$$

The second term due to the orbital eccentricity may be expressed

$$\Delta t_{rel} = \frac{2}{c^2} \sqrt{GM\,a}\, e \sin E = \frac{2\,\mathbf{r} \cdot \mathbf{v}}{c^2}$$

Electromagnetic signals from Earth-orbiting clocks

In the rotating ECEF coordinate system, the metric components are

$-g_{00} \approx 1$, $g_{0j} = (\omega \times \mathbf{r})_j / c$, and $g_{ij} \approx \delta_{ij}$

The coordinate time of propagation of an electromagnetic signal is

$$\Delta t = \frac{\rho}{c} + \frac{1}{c} \int_{\text{path}} \sum_{j=1}^{3} \frac{g_{0j}}{-g_{00}} dx^j$$

The integral term is called the Sagnac effect, which may be expressed.

$$\Delta t_{\text{Sagnac}} \approx \frac{1}{c^2} \int_A^B (\boldsymbol{\omega} \times \mathbf{r}) \cdot d\mathbf{r} = \frac{1}{c^2} \int_A^B \boldsymbol{\omega} \cdot (\mathbf{r} \times d\mathbf{r}) = 2 \frac{1}{c^2} \int_A^B \boldsymbol{\omega} \cdot d\mathbf{A} = \frac{2\omega A}{c^2}$$

For endpoints at (x_A , y_A) and (x_B , y_B), the Sagnac effect is

$$\Delta t_{\text{Sagnac}} = \frac{2\omega A}{c^2} = \frac{\omega}{c^2} (x_A y_B - y_A x_B)$$

The Global Positioning System (GPS)

The GPS provides a paradigm for the application of the principles of general relativity to a practical engineering system.

The fundamental measurement is pseudorange (PR), which is the phase shift necessary to bring about correlation between a pseudorandom noise (PRN) signal transmitted by the satellite and a replica PRN signal generated in the receiver.

The PR measurements must be corrected for ionospheric delay, tropospheric delay, satellite clock offset, and relativity. Four PR measurements are required to solve for x, y, z, and ΔT.

Relativistic effects in the GPS

For measurements with a precision at the one-to-ten nanosecond level, there are ***three relativistic effects*** that must be considered.

First, there is the effect of ***time dilation***. GPS satellites revolve around the Earth with an orbital period of 11.967 hours and a velocity of 3.874 km/s. On account of its velocity, a GPS satellite clock appears to run slow by 7 μs per day.

Second, there is the effect of the ***gravitational redshift***. At an altitude of 20 184 km, the clock runs fast by 45 μs per day.

The ***net effect of time dilation and gravitational redshift*** is that the satellite clock runs fast by approximately 38 μs per day when compared to a similar clock at rest on the geoid, including the effects of the Earth's rotation and the gravitational potential at the Earth's surface.

With an orbital eccentricity of 0.02, there is a relativistic sinusoidal variation in the apparent clock time having an amplitude of 46 ns at the orbital period.

The third relativistic effect is the ***Sagnac effect***. For a stationary terrestrial receiver on the geoid, the Sagnac correction can be as large as 133 ns.

Relativistic Effects in Earth-Orbiting Satellite Clocks

Constants							
Velocity of light	m/s	299 792 458					
Gravitational constant of Earth	km^3/s^2	398 600.44					
Radius of Earth	km	6378.137					
J_2 oblateness coefficient		0.0010826					
Angular velocity of Earth rotation	rad/s	7.292×10^{-5}					
Geopotential on geoid U_0	m^2/s^2	6.264×10^{7}					
U_0/c^2		-6.969×10^{-10}					
Satellite orbital properties							
Satellite		**ISS**	**GLONASS**	**GPS**	**Galileo**	**Molniya**	**GEO**
Semimajor axis	km	6766	25510	26561.8	29994	26562	42164
Eccentricity		0.01	0.02	0.02	0.02	0.722	0.01
Inclination	deg	51.6	64.8	55.0	56.0	63.4	0.1
Argument of perigee	deg	0	0	0	0	250	0
Apogee altitude	km	456	19642	20715	24216	39362	36208
Perigee altitude	km	320	18622	19652	23016	1006	35364
Ascending node altitude	km	320	18622	19652	23016	10507	35364
Period of revolution	s	5539	40549	43082	51697	43083	86164
Mean motion	rev/d	15.6	2.1	2.0	1.7	2.0	1.0
Mean velocity	km/s	7.675	3.953	3.874	3.645	3.874	3.075
Clock effects							
Secular time dilation	μs/d	-28.2	-7.4	-7.1	-6.3	-7.1	-4.4
Secular redshift	μs/d	3.5	45.1	45.7	47.3	45.7	51.0
Net secular effect	μs/d	-24.7	37.7	38.6	41.1	38.6	46.6
Amplitude of periodic effect due to eccentricity	ns	12	45	46	49	1653	29
Secular oblateness contribution to redshift	ns/d	23.7	0.8	0.5	0.4	2.5	-0.1
Amplitude of periodic effect due to oblateness	ps	264	50	38	33	167	0
Amplitude of periodic tidal effect of Moon	ps	0.0	1.0	1.2	1.8	1.2	6.1
Amplitude of periodic tidal effect of Sun	ps	0.0	0.5	0.5	0.8	0.5	2.7
Signal propagation							
Maximum Sagnac effect	ns	13	131	136	155	234	218
Gravitational propagation delay along radius	ps	0.8	-3.5	-4.7	-9.1	-4.7	-27.3
Amplitude of periodic fractional Doppler shift	10^{-12}	13.1	7.0	6.7	5.9	241.1	2.1

Time transfer between Mars and Earth

Proper time as measured by clocks on Mars surface

Proper time as measured by clock on Mars spacecraft

Mars Spacecraft

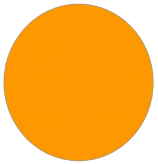
Mars

Barycentric Coordinate Time (TCB)

Earth

GPS Satellite

Proper time as measured by clock on GPS satellite



Sun

Proper time as measured by clocks on Earth's surface

Terrestrial Time (TT)

International Atomic Time (TAI)

Coordinated Universal Time (UTC)

GPS Time

Terrestrial Time (TT) and Barycentric Coordinate Time (TCB)

Terrestrial Time (TT) is a proper time realized by clocks on the geoid. A practical realization of TT in terms of International Atomic Time (TAI) is

$$\mathrm{TT} = \mathrm{TAI} + 32.184\,\mathrm{s}$$

The transformation from TT to Geocentric Coordinate Time (TCG) is

$$\mathrm{TCG} - \mathrm{TT} = (W_{0E}/c^2)\,\Delta T = L_G\,\Delta T$$

where W_{0E} is Earth geopotential, $L_G \equiv W_{0E}\ / c^2 = 6.969\,290\,134 \times 10^{-10} \cong 60.2\ \mu\mathrm{s/d}$, and ΔT is the time elapsed since 1 January 1977 0 h TAI (JD 244 3144.5).

Barycentric Coordinate Time (TCB) is the coordinate time Δt in a barycentric coordinate system. The coordinate time Δt corresponding to the proper time $\Delta\tau$ maintained by a clock on the geoid is

$$\Delta t = \int_{\tau_0}^{\tau} \left[1 + \frac{1}{c^2}U(\mathbf{r}) + \frac{1}{2}\frac{1}{c^2}v^2\right] d\tau$$

Modification of integral for coordinate time

s desirable to separate the clock–dependent part from the clock–independent pa

$$\mathbf{r} = \mathbf{r}_E + \mathbf{R}$$

$$\mathbf{v} = \mathbf{v}_E + \dot{\mathbf{R}}$$

$$U(\mathbf{r}) = U_E(\mathbf{r}) + U_{\text{ext}}(\mathbf{r})$$

$$U_{\text{ext}}(\mathbf{r}) \approx U_{\text{ext}}(\mathbf{r}_E) + \nabla U_{\text{ext}} \cdot \mathbf{R}$$

$$v^2 = v_E^2 + 2\dot{\mathbf{R}} \cdot \mathbf{v}_E + |\dot{\mathbf{R}}|^2$$

$$\dot{\mathbf{R}} \cdot \mathbf{v}_E = \frac{d}{dt}(\mathbf{R} \cdot \mathbf{v}_E) - \mathbf{R} \cdot \mathbf{a}_E$$

$$\mathbf{a}_E \equiv \frac{d\mathbf{v}_E}{dt} = \nabla U_{\text{ext}}$$

Thus the elapsed coordinate time is

$$\Delta t \approx \Delta\tau + \frac{1}{c^2}\left[\int_{t_0}^{t} U_{\text{ext}}(\mathbf{r}_E) + \frac{1}{2}v_E^2\right] dt + \frac{1}{c^2}\left[\int_{t_0}^{t} U_E(\mathbf{R}) + \frac{1}{2}|\dot{\mathbf{R}}|^2\right] dt + \frac{1}{c^2}\mathbf{R} \cdot \mathbf{v}_E \Big|_{t_0}^{t}$$

Barycentric Coordinate Time - Terrestrial Time

The proper time $\Delta\tau$ is identified with Terrestrial Time (TT). The coordinate time Δt is identified with Barycentric Coordinate Time (TCB). Thus

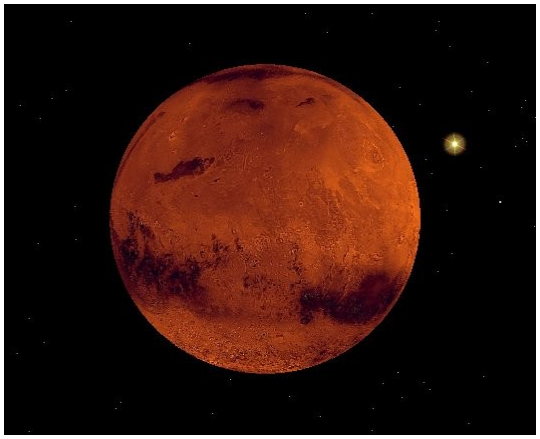
$$\mathrm{TCB}-\mathrm{TT}=\frac{1}{c^2}\int_{t_0}^{t}\left[U_{\mathrm{ext}}(\mathbf{r}_E)+\frac{1}{2}v_E^2\right]dt+L_G\,\Delta T+\frac{1}{c^2}\mathbf{R}\cdot\mathbf{v}_E\Big|_{t_0}^{t}=L_C\,\Delta T+P+L_G\,\Delta T+\frac{1}{c^2}\mathbf{R}\cdot\mathbf{v}_E\Big|_{t_0}^{t}$$

where $L_C = 1.480\,826\,867\,41\times10^{-8} \cong 1.28$ ms/d and ΔT is the time elapsed since 1 January 1977 0 h TAI (JD 2443144.5). For a clock on the geoid, the diurnal term has a maximum amplitude of 2.1 μs (for a clock on the equator). The leading terms in the evaluation of the integral are

$$\frac{1}{c^2}\int_{t_0}^{t}\left[U_{E\,\mathrm{ext}}(\mathbf{r}_E)+\frac{1}{2}v_E^2\right]dt\approx\frac{3}{2}\frac{1}{c^2}\frac{GM_S}{a_E}\Delta T+\frac{2}{c^2}\sqrt{GM_S\,a_E}\;e_E\sin E_E$$

Barycentric Coordinate Time - Mars Time

Similar transformations apply to the transfer of time from the surface of Mars to the solar system barycenter.



he transformation from Mars Time (MT) to Areocentric Coordinate Time (TCA) is

$$\text{TCA} - \text{MT} = (W_{0M} / c^2)\, \Delta T = L_M$$

ΔT

Then the transformation between TCB and MT is

$$\text{TCB} - \text{MT} = \frac{1}{c^2}\int_{t_0}^{t}\left[U_{\text{ext}}(\mathbf{r}_M) + \frac{1}{2}v_M^2\right]dt + L_M\,\Delta T + \frac{1}{c^2}\mathbf{R}\cdot\mathbf{v}_M\Big|_{t_0}^{t} = L_{CM}\,\Delta T + P + L_M\,\Delta T + \frac{1}{c^2}\mathbf{R}\cdot\mathbf{v}_M\Big|_{t_0}^{t}$$

where

$$\frac{1}{c^2}\int_{t_0}^{t}\left[U_{M\,ext}(\mathbf{r}_M) + \frac{1}{2}v_M^2\right]dt \approx \frac{3}{2}\frac{1}{c^2}\frac{GM_S}{a_M}\Delta T + \frac{2}{c^2}\sqrt{GM_S\,a_M}\;e_M\sin E_M$$

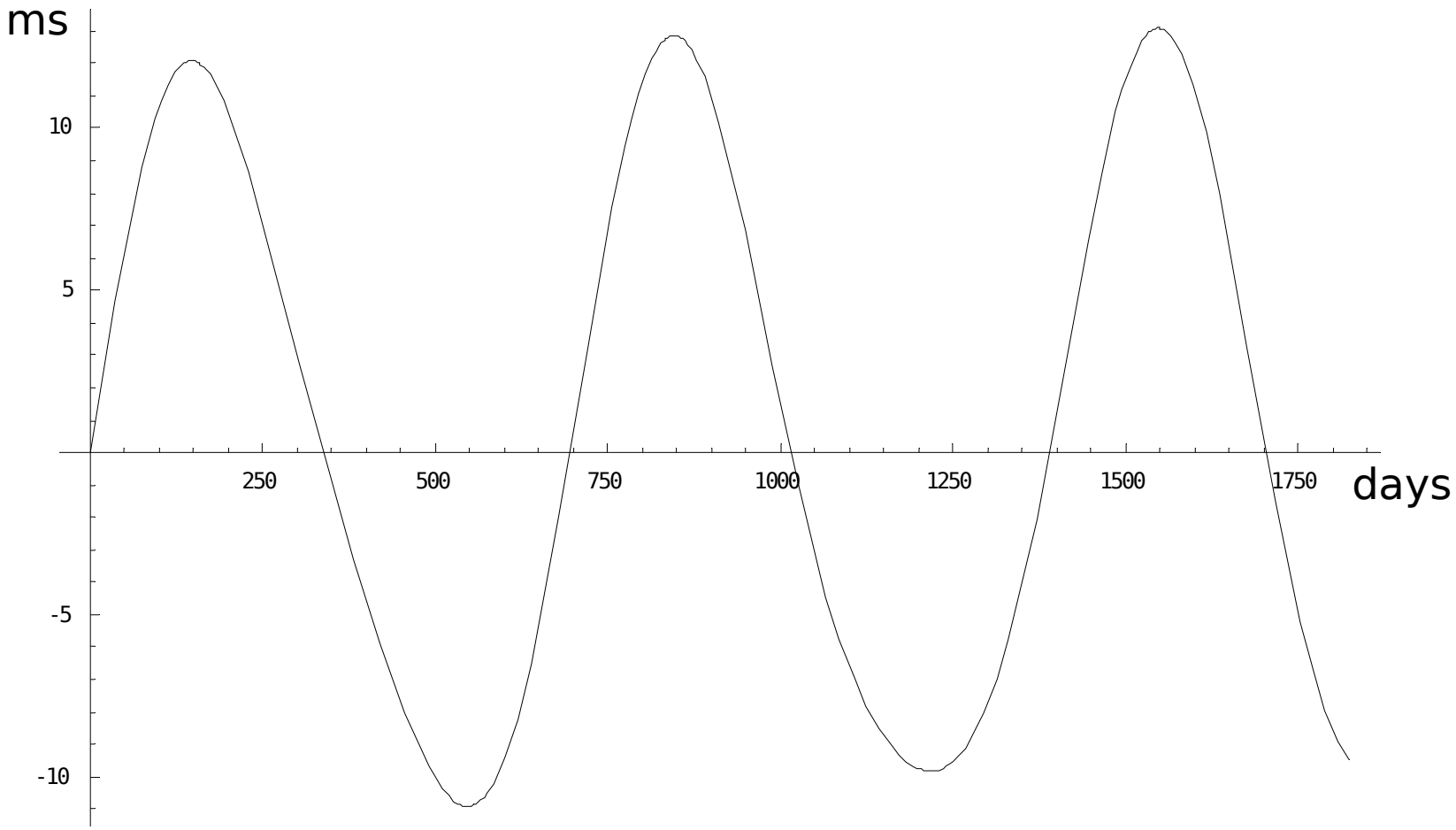
Summary of relativistic effects

Geoid to geocenter	
Secular drift	60.2 μs/d
Maximum amplitude of diurnal term	2.1 μs
Geocenter to Barycenter	
Secular drift	1.28 ms/d
Amplitude of principal periodic term	1.7 ms
Mars surface to Mars center	
Secular drift	12.1 μs/d
Maximum amplitude of diurnal term	0.9 μs
Mars center to Barycenter	
Secular drift	0.84 ms/d
Amplitude of principal periodic term	11.4 ms

$$MT - TT = (TCB - TT) - (TCB - MT)$$

net secular drift is (1.28 ms/d + 0.06 ms/d) – (0.84 ms/d + 0.01 ms/d) = 0.49 m

Net relativistic periodic effect for time transfer between Mars and Earth



Relativistic Transformation from the Moon to Earth

For time transfer from the surface of the Moon to the surface of the Earth, the procedure is similar but the relative magnitudes of the terms is different. A convenient coordinate system is one whose origin is at the center of the Earth. (The motion of the Earth's center about the center of mass of the Earth-Moon system will be neglected.) The difference between Geocentric Coordinate Time and Terrestrial Time, the proper time measured by clocks on the Earth's surface, is

$$\mathrm{TCG} - \mathrm{TT} = L_G \, \Delta T$$

Also, the difference between Geocentric Coordinate Time (TCG) and Lunar Time (LT), the proper time measured by clocks on the Moon's surface, is

$$\mathrm{TCB} - \mathrm{LT} = \frac{1}{c^2} \int_{t_0}^{t} \left[U_{\mathrm{ext}}(\mathbf{r}_m) + \frac{1}{2} v_m^2 \right] dt + L_m \Delta T + \frac{1}{c^2} \mathbf{R} \cdot \mathbf{v}_m \Big|_{t_0}^{t} = L_{Cm} \Delta T + P + L_m \Delta T + \frac{1}{c^2} \mathbf{R} \cdot \mathbf{v}_m \Big|_{t_0}^{t}$$

$$L_{Cm} \approx \frac{3}{2} \frac{1}{c^2} \frac{GM_E}{a_m} \qquad P \approx \frac{2}{c^2} \sqrt{GM_E \, a_m} \, e_m \sin E_m \qquad L_m \approx \frac{1}{c^2} \frac{GM_m}{R_m}$$

LT – TT = (TCG – TT) – (TCG – LT)

Thus the net secular drift rate is 60.2 μs/d – (1.5 μs/d + 2.7 μs/d) = 56.0 μs/d and the amplitude of the periodic effect is 0.48 μs at the Moon's orbital period (27.3 d).

Conclusion

In the future exploration of the solar system, there will be a need for coherent time synchronization and dissemination. This will require the application of appropriate relativistic algorithms in a common coordinate system (such as the barycentric coordinate system).

The effects of relativity are by no means negligible. They have been successfully demonstrated in the Global Positioning System (GPS). Analogous algorithms will be required in the future exploration of the Moon, Mars, and beyond.

The amplitude of the relativistic difference between the proper time reading of a clock on Mars and the proper time reading of a clock on Earth is up to 13.1 ms. Neglect of this effect using GPS-like signals transmitted by clocks on Mars and an ephemeris expressed in terms of Terrestrial Time would cause a navigation error of about 3000 km.

Appendix

Planetary Reference Data

Reference Data for the Earth and Mars

Mass		
	Sun	1.9891×10^{30} kg
	Earth	5.9742×10^{24} kg
	Mars	0.6419×10^{24} kg
Planetary radius		
	Earth	6378 km
	Mars	3397 km
Orbital semimajor axis		
	Earth	1.000 AU = 1.496×10^{8} km
	Mars	1.524 AU = 2.279×10^{8} km
Orbital period		
	Earth	365.2422 d
	Mars	686.9297 d
Average orbital velocity		
	Earth	29.8 km/s
	Mars	24.1 km/s
Orbital eccentricity		
	Earth	0.0167
	Mars	0.0934
Speed of light		299,792,458 m/s (exact)
Gravitational constant		$6.6726 \times 10^{-11} m^3$ / kg s^2

Reference Data for the Moon

Mass	0.07353×10^{24} kg
Radius	1738.2 km
Orbital semimajor axis	384,400 km
Orbital eccentricity	0.05490
Average orbital velocity	1.023 km/s
Distance of geocenter from barycenter	4671 km